Mathematics Explanations: Sections 2 and 4

Section 2: Mathematics

1) C) To calculate how many lawns can be mowed in 8 hours, set up a proportion. To make sure you set it up right, make sure the units line up on both sides. In other words, this/that = this/that. Here, you have (lawns)/(hours) = (lawns)/(hours), so 45 lawns/12 hours = x lawns/8 hours x = 8 • 45/12 = 30.

2) C) If PQ is twice as long as QR, which is 6 units long, PQ is 12 units long. Therefore, PR = 12 + 6 = 18.

3) E) Since all of the numerators are even, they all must yield integers when divided by 2. 6/2 = 3, 12/2 = 6, and 4/2 = 2.

4) B) Remember that probability can be calculated as (# of desired outcomes)/(# of possible outcomes). After the green marble is chosen, there are three possible outcomes since there are three marbles left in the jar. Only one of these three, the blue marble, is a desired outcome, so the probability of drawing a blue marble is 1/3. If you chose A, you might have missed the sentence stating that the green marble is already removed.

5) A) Based on parallel line properties (specifically alternating interior angles), the missing angle of the bottom triangle must be 30˚. Because the angles of a triangle add up to 180˚, 20 + 30 + z = 180, so z = 130.

6) B) The slow way to do this problem is to solve for x using the first equation. 4x = 16 x = 4. Now substitute 4 for x in the second expression. 2(4) – 1 = 7. The fast way to do the problem is to recognize that 2x – 1 is equal to half the expression on the left side of the first equation, so it must be equal to half of 14, which is 7.

7) D) Model the price of both plans using linear equations. When modeling a linear situation, the y-intercept (b in y = mx + b) is equal to the fixed amount and the slope (m in y = mx + b) is equal to the variable amount. In this case, the y-intercept is equal to the per day cost (since the car is only being rented for one day), and the slope is equal to the cost per mile. Therefore, for plan 1, c = .3m + 20 and for plan 2, c = .35m + 10, where c is the cost and m is the mileage driven. Set the right sides of these equations equal to find the value of m for which the cost is the same. .3m + 20 = .35m + 10 10 = .05m m = 200 miles.

8) D) Since point O is located at the origin (0, 0) and point A is located at (1, 2), point A is up 2 units and to the right 1 unit from point O. Because the slope of a line (the ratio of its rise to its run) is constant, the point that is 1 up and 2 over from A would also be on this line, and this point is (2, 4). Alternatively, you could have used the slope formula (rise/run = (change in y)/(change in x) = (y2 – y1)/(x2 – x1)) to calculate that the slope of the line is 2. Then you could check each answer using this same formula to see if it creates a slope of 2 with either point A or point O. However, this method is unnecessarily time consuming.

9) A) Because 40% of cable subscribers subscribe to Invent Cable, 60% of cable subscribers subscribe to other services. Therefore, the answer can be found by finding 60% of the 5,000,000 cable subscribers, so .6 • 5,000,000 = 3,000,000.

10) A) Do guess and check by plugging the answers in for b and seeing if the expression yields the corresponding value of g(b) from the table. Remember that the bars around the 2b are absolute value bars, which make the value positive whether 2b is positive or negative to begin with. Based on the method above, A is the only answer that works, since | | + 1 = | | + 1 = 6 + 1 = 7, which is the value of g(b) in the table when b = -3.

11) B) Simplify the equation in the question to get 3x = -m. Then plug in 3 for m to get 3x = -3, so x = -3/3 = -1. If you answered A, you may have miscalculated the right side of the equation as -3m instead of -m.

12) A) "Twice" means 2 *, "sum" means addition, "decreased" means minus, and "product" means multiplication. Use these definitions to slowly translate the English in the statement into a mathematical expression, and you should get A. The most tempting incorrect answer is C, which is incorrect because the language in the statement makes it clear that x and y must be added together and then multiplied by 2.

13) B) There are six faces of a cube, and the surface area of a cube is the sum of the areas of all six of these faces, so the area of 1 face is equal to the surface area of the cube divided by 6. 48/6 = 8. If you answered E, you may have mistakenly thought a cube had four faces.

14) E) To minimize the value of the fraction in the question, you must maximize the denominator, √ . √ will be largest when x itself is its largest, so E, the largest answer, is correct. By no means is it necessary to calculate the actual value of the fraction for each of the answer choices. Instead, simply choose the largest answer choice offered based on the above reasoning.

15) C) Since CE = CD, triangle CED is isosceles. The base angles of an isosceles triangle (the angles opposite the two equal sides) are always equal, so because x = 70, angle D = 70˚. Therefore angle ECD = 180˚ – 70˚ – 70˚ = 40˚. Because vertical angles are equal, angle BCA =40˚. Since AB = BC, triangle ABC is isosceles. Therefore, its base angles, angle BCA and angle A must be equal, so angle A = 40˚. Because the angles of a triangle add up to 180˚, angle ABC = 180˚ – 40˚ – 40˚ = 100˚.

16) C) A parabola is symmetric about a vertical line drawn through its vertex, which for this parabola is the line x = -2. Therefore, since (-5, 3) is on the parabola, its reflection across the line x = -2 must also be on the parabola. Since -5 is 3 less than -2, the point of the reflection must have an x-value 3 more than -2, so its x-value must be 1. Its y-value must be the same as the y-value of (-5, 3), so its y-value must be 3. Therefore, the point (1, 3) must also be on this parabola.

17) D) Because there are a total of 91 numbers, you know that x + y = 91 – 15 – 19 – 11, so x + y = 46. The mode of a set is the most frequently occurring number in the set, so if 80 is the mode of the set, its frequency must be greater than that of the next greatest frequency, so x > 19. You also know that y < x, or 88 would be the mode of the set. Since y < x and since x + y = 46, which is an even number, the largest possible value of y will be found when it is 2 less than x. (If y were only 1 less than x, x + y would have to be odd and could therefore not be equal to 46). So let y = x – 2. Now substitute x – 2 for y in the equation x + y = 46 to get x + (x –2) = 46 2x – 2 = 46 2x = 48 x = 24. If x = 24, then 24 + y = 46, so y = 22. It turned out to be irrelevant that the median (the middle number) of the set is 88, because when x = 24 and y = 22, the median of the set is indeed 88.

18) D) The distance from 1 to x is equal to their difference, x – 1. Because this distance is one tick mark and there are two tick marks between x and y, the distance between x and y is equal to 2(x – 1) = 2x – 2.

Since the distance between x and y is also equal to their difference, y – x, you can set y – x equal to 2x – 2 and solve for y. y – x = 2x – 2 y = 3x – 2.

19) E) In counting problems, you should consider how many events are occurring and how many ways each event can occur. Here you have 4 events since the dog goes through four doors. It might be wise at this point to write out slots for each event with multiplication symbols in between them: _•_•_•_. Then, populate these slots with how many ways each event can occur. Here, the dog has 2 door options from Room T to Room S and 3 options from Room S to Room R. Because the dog cannot reuse the same door, he then has 2 door options from Room R to Room S and 1 door option from Room S to Room T. Populate the slots drawn above with these numbers to get 2 • 3 • 2 • 1 = 12.

20) D) I is not true for all values because it is false for any negative value of x, since negative numbers squared are positive. II is true for all values because if (x – 2) is positive or negative, (x – 2)2 is positive, and if (x – 2) is equal to zero, then (x – 2)2 is also equal to zero. III is true for all values because the two expressions are equal when x = 1, and the left side is greater than the right side for all other values of x. One easy way to figure out which inequalities are true for all values is to graph the left side as y1 and the right side as y2 on your graphing calculator. If the graph of y1 is always above (has a greater y-value) or equal to the graph of y2 at every x-value, then the inequality is true.

Section 4: Mathematics

21) B) Substitute 3 for xy in the first equation to get (6)(3)z = 72 18z = 72 z = 72/18 = 4.

22) E) The graph shows that 2 students had 8 correct responses, 3 students had 9 correct responses, and 1 student had 10 correct responses, so 2 + 3 + 1 = 6 students had more than 7 correct responses.

23) D) 36 is the square of 6 since 62 = 36, and 36 is a multiple of 3 since 3 • 12 = 36. Answer choices A and D are not squares of integers, and answer choices B and C are not multiples of 3.

24) A) Since PQST and VRSU have the same dimensions and RS = 2, ST = 2 as well. Because the area of a rectangle is equal to its length times its width, if the area of PQST is 14 and ST = 2, then QS = 14/2 = 7. If QS = 7 and RS = 2, then QR = 7 – 2 = 5.

25) B) Distribute the 3 to both terms inside the parentheses in both equations to get y = 3x + 6 and z = 3x + 9. 3x + 9 is 3 greater than 3x + 6, so y is 3 greater than z.

26) A) The triangle inequality states that the sum of the lengths of two sides of a triangle must be greater than the length of the third side. Based on this property, if you know the lengths of two sides of a triangle and are trying to find the range of lengths of the third side, you can add the two known side lengths together and subtract the smaller one from the bigger one. The third side must be greater than the sum of the other two sides and less than their difference. In this case, AC > 7 – 5, so AC > 2, and AC < 7 + 5, so AC < 12. Therefore, all the answer choices are possible lengths of AC except for 1.

27) C) If the first three picks produce one red sock, one white sock, and one blue sock (the only possible way there cannot already be two of the same color among the first three selected), then the fourth pick will necessarily be the same color as one of the three already picked, since the fourth pick must be red, white, or blue.

28) C) angle BEA and angle CED are both supplementary to angle AED, which is 140˚. Therefore, angle BEA and angle CED are both equal to 180˚ – 140˚ = 40˚. Since there is a total of 360˚ in a circle, each of these 40 angles makes up 40/360 = 1/9 of the circle, so the arcs these angles intercept, minor arcs AB and CD, must each have a length equal of 1/9 of the circle's circumference. Because the circle has a radius of 9, its circumference is equal to 18π. Therefore, minor arcs AB and CD must each have a length of 1/9 • 18π = 2π, so the sum of their lengths is 4π.

29) 40) Since the three angles of a triangle must add up to 180˚, x + 3x + 20 = 180 4x = 160 x = 40.

30) 1080) If 10 percent read no novels, the remaining 90 percent read at least one. Therefore, the answer can be found by finding 90% of 1200. .9 • 1200 = 1080. Alternatively, you could have found this answer by finding 10 percent of 1200 and subtracting this number from 1200.

31) 16) The first equation can be simplified by adding the exponents to get y = 2a+b. Then you can substitute 4 for a + b to get y = 24 = 16.

32) 600) To solve for k, substitute 500 for C(t) and 2 for t in the equation to get 500 = 100(2 + k) 500 = 200 + 100k 300 = 100k k = 3. Now substitute 3 for k and 3 for t in the original equation and solve for C(t). C(3) = 100(3 + 3) = 100(6) = 600. Alternatively, you could have solved this problem without doing any of the above work just by recognizing that, since k is a constant, C(t) will increase by 100 for every 1 unit increase in t, so if C = 500 when t = 2, c = 600 when t = 3.

33) 8) This problem is easiest when you draw it out. Draw a 6x6 square and divide its length into 3 by drawing two horizontal lines, and divide its width into 3 by drawing 2 vertical lines. These lines you have drawn divide the square into 9 smaller squares. Assuming that these 9 smaller squares are equal, each one has a side length of 2, since the length and width of the original square are divided into 3 equal portions and 6/3 = 2. Therefore, the perimeter of one of the smaller squares is 2 • 4 = 8. If you answered 4, you may have found the area of one of the smaller squares.

34) 108) A prime number is a number that has no factors other than itself and 1. The largest prime number less than 100 is 97, since the only two factors of 97 are 1 and 97. The smallest prime number greater than 10 is 11, since the only two factors of 11 are 1 and 11. Therefore a + b = 97 + 11 = 108.

35) .833 or 5/6) Apply the changes in the table beginning with a depth of 400. To find the depth at the beginning of 2000, include the changes through 1999: 400 + 150 – 125 + 75 – 125 = 375. Then add 75, the change in the year 2000, to 375 to get 450, the depth at the end of 2000. Therefore, the answer to the question is equal to the depth at the beginning of 2000 divided by the depth at the end of 2000: 375/450. Simplify this fraction to get 5/6.

36) 36) Any sequence where each term is multiplied by the same number to produce the next term is known as a geometric sequence, and the number each term is multiplied by to produce the next term is known as the common ratio. This number is called the common ratio because if you divide any term by the previous term (in other words, find the ratio of any term to the term before it), it will equal this number. In this case, the common ratio is k. As described above, the value of k can be found by dividing any term by the previous term, so k = 12/2 = 6. If the common ratio is 6, then the 52nd term divided by the 51st term must equal 6, and the 51st term divided by the 50th term must equal 6, so the 52nd term divided by the 50th term must equal 6 • 6 = 36. Indeed, any term in this sequence, when divided by the term two before it, will produce a value of 36.

37) .75 or 3/4) If the average of the two numbers is 1, then the two numbers must be the same amount above and below 1. If the difference of the two numbers is 1, then they are 1 apart from each other, so one must be .5 above 1 and the other must be .5 below 1, so the two numbers are .5 and 1.5. A product is the result of multiplication, so multiply .5 and 1.5 to get the answer: $.5 \cdot 1.5 = .75$. Alternatively, you could have done this problem as a linear system. Since the average of the two numbers is 1, you could set up the following equation: $(x + y)/2 = 1$, which simplifies to $x + y = 2$. Because their difference is 1, you could set up the following equation: $x - y = 1$, which simplifies to $x = y + 1$. If $x = y + 1$, you can substitute $(y + 1)$ for x in the first equation to get $(y + 1) + y = 2$ $2y + 1 = 2$ $2y = 1$ $y = .5$. Substitute .5 for y in either equation to get $x = 1.5$, and then multiply .5 by 1.5 to get your answer.

38) $3/2 < x < 5/3$) The area of a triangle is equal to $(1/2)bh$, where b is the base and h is the height, which is by definition perpendicular to the base. Let AB be the base of this triangle. The length of $AB = p - -p = 2p$. If AB is the triangle's base, the triangle's height is the segment of the x-axis between the origin and the point where the x-axis intersects AB. Since x-coordinate of both A and B is 6 and point O has an x-coordinate of 0, the length of this height is equal to $6 - 0 = 6$. Therefore, the area of this triangle is equal to $(1/2)(6)(2p) = 6p$. Therefore, $6p > 9$, so $p > 9/6$ which simplifies to $p > 3/2$, and $6p < 10$, so $p < 10/6$ which simplifies to $p < 5/3$. Therefore any answer between 3/2 and 5/3 (but not equal to 3/2 or 5/3) would be acceptable.